



TERRAMETRA

Trigonometry
“Plane” and Simple to “Spherical”

Terrametra Resources

Lynn Patten



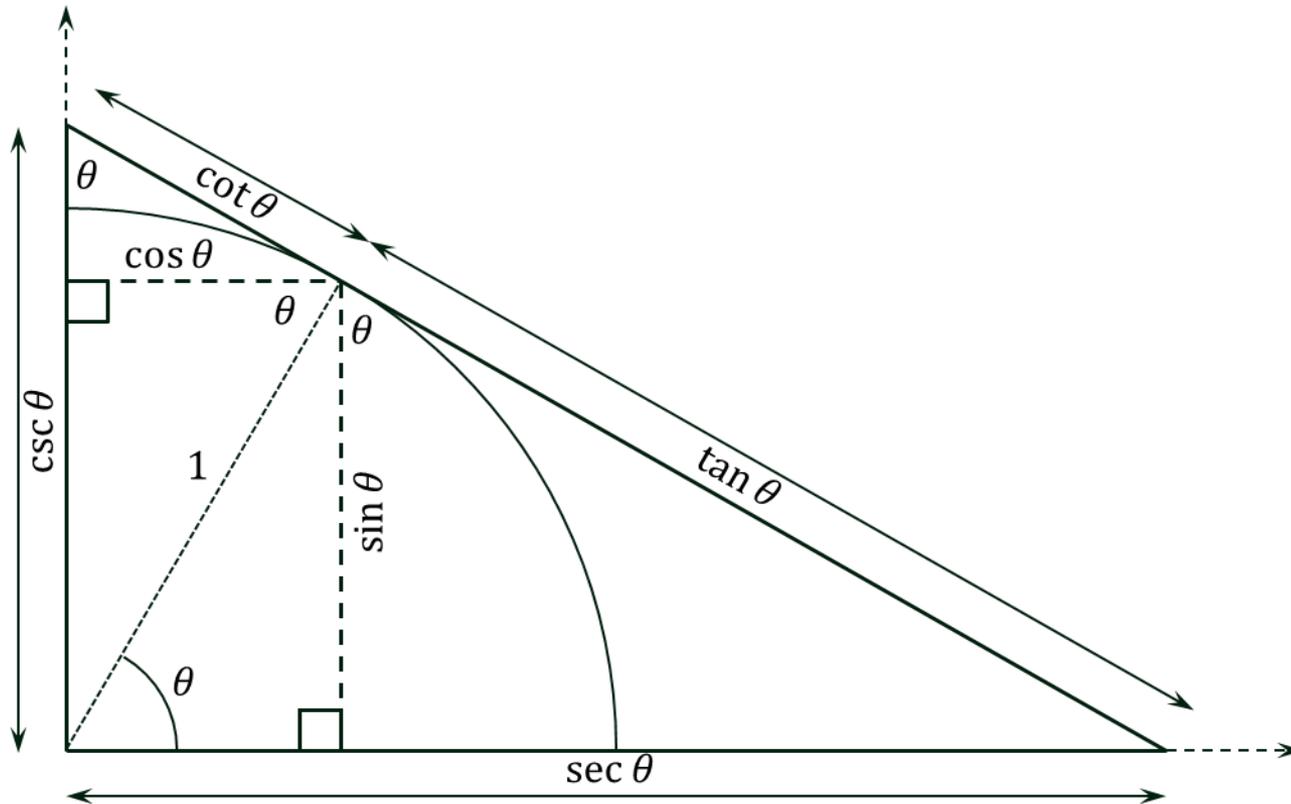
Trigonometric Identities

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The Trigonometric Functions



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

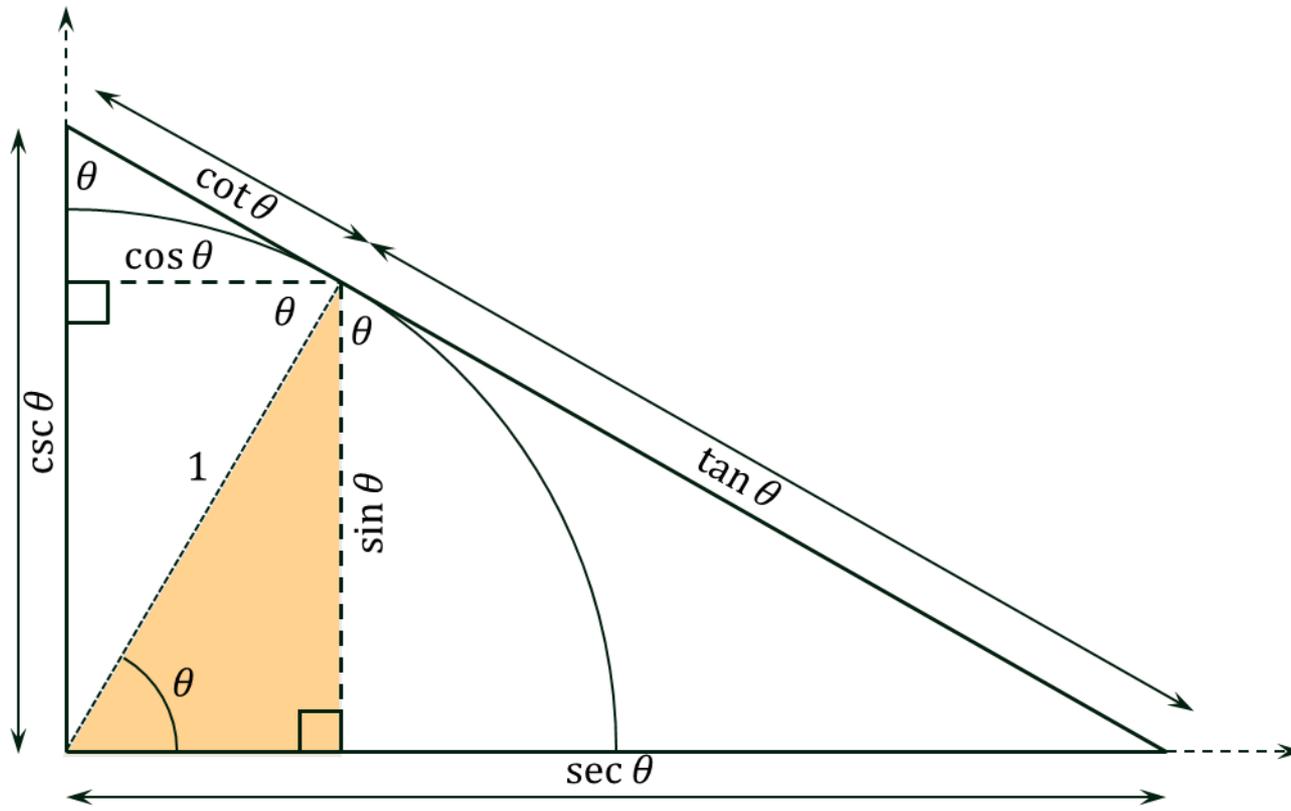
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



The SIN Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

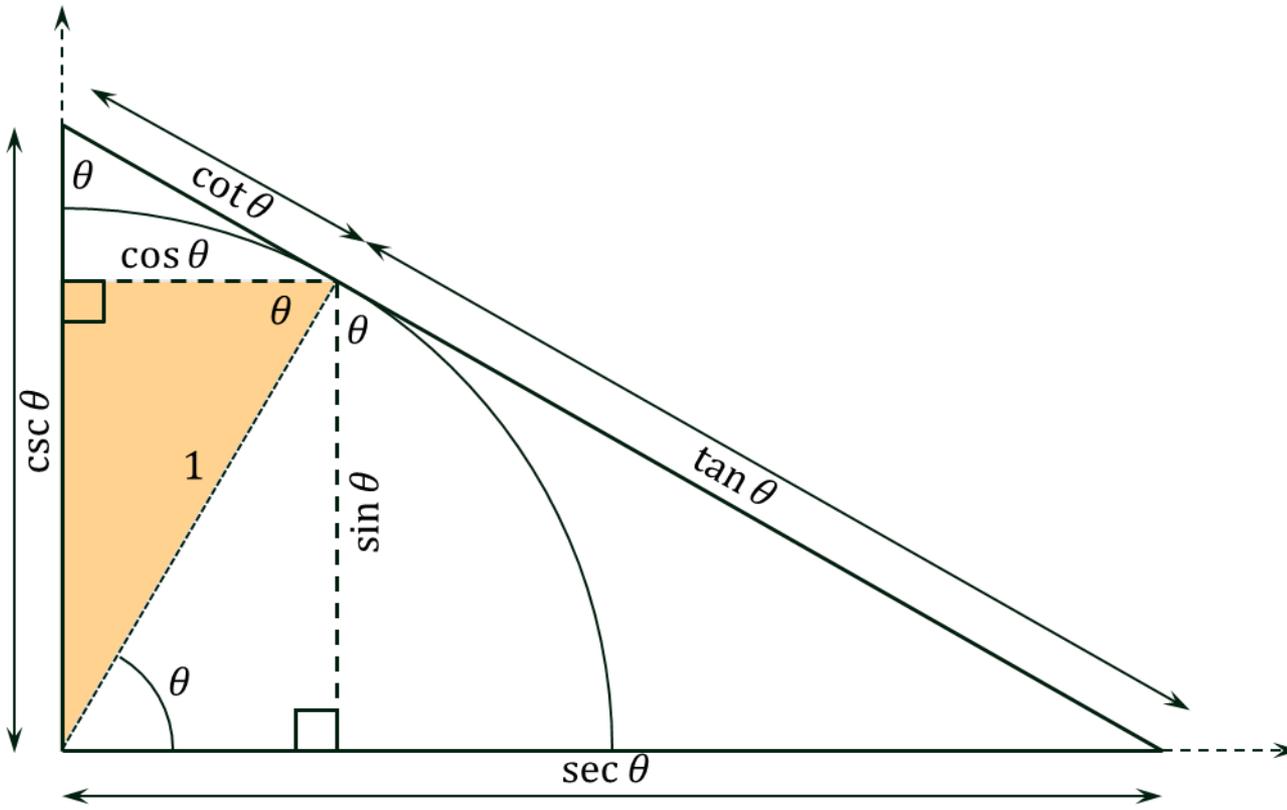
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



The COS Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

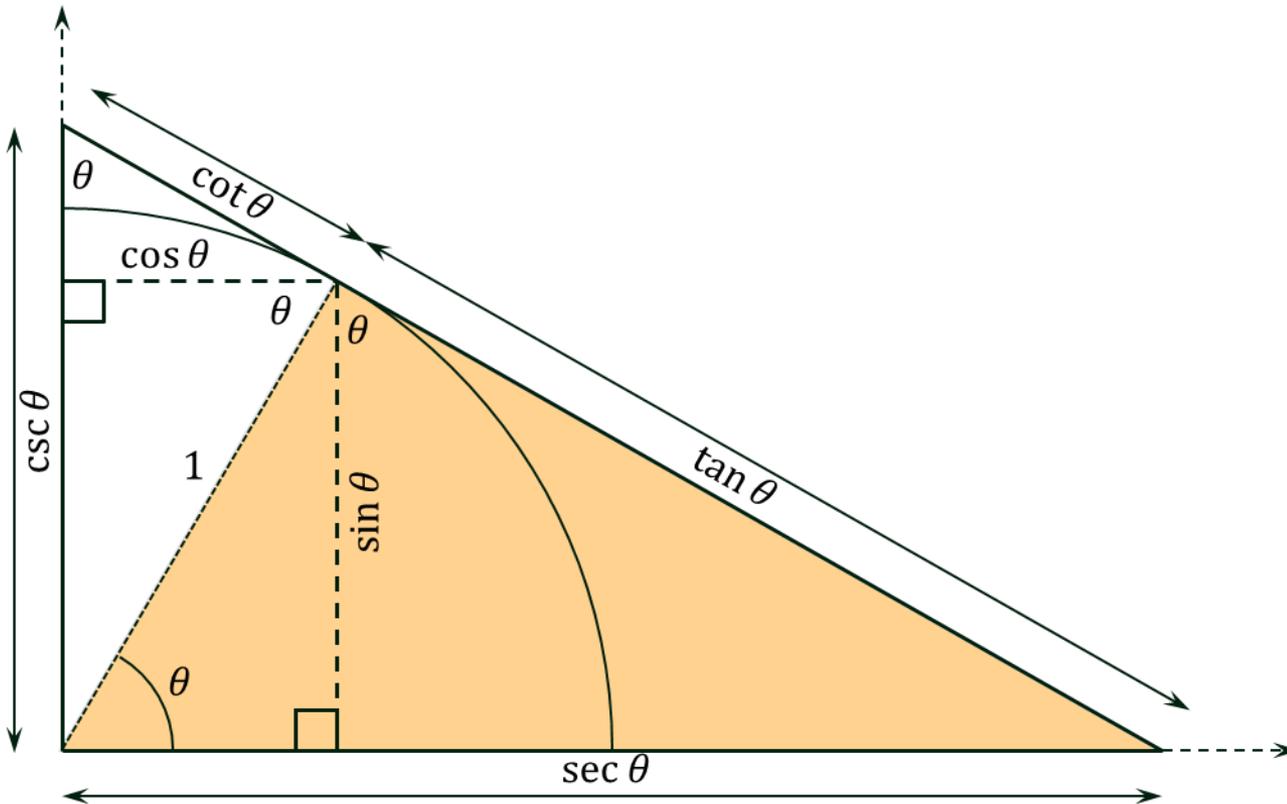
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



The TAN Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

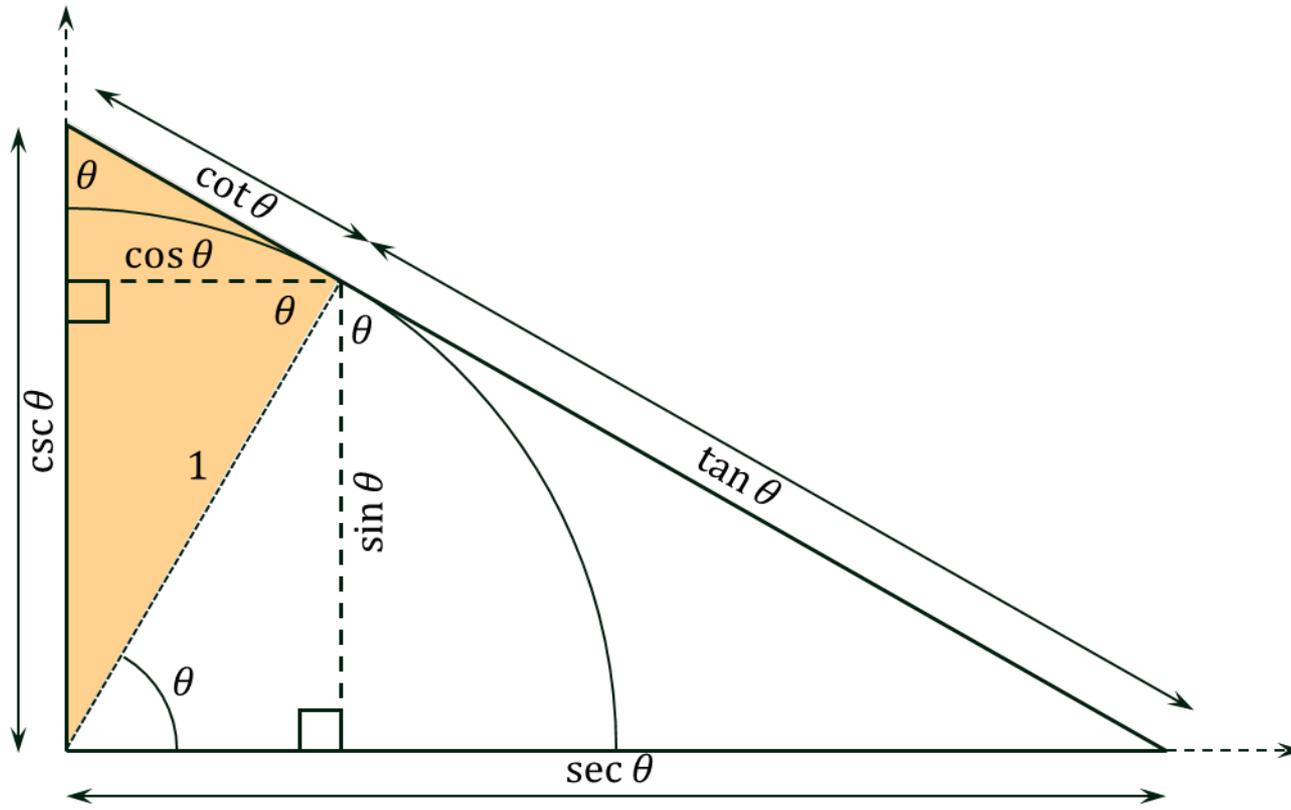
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



The COT Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

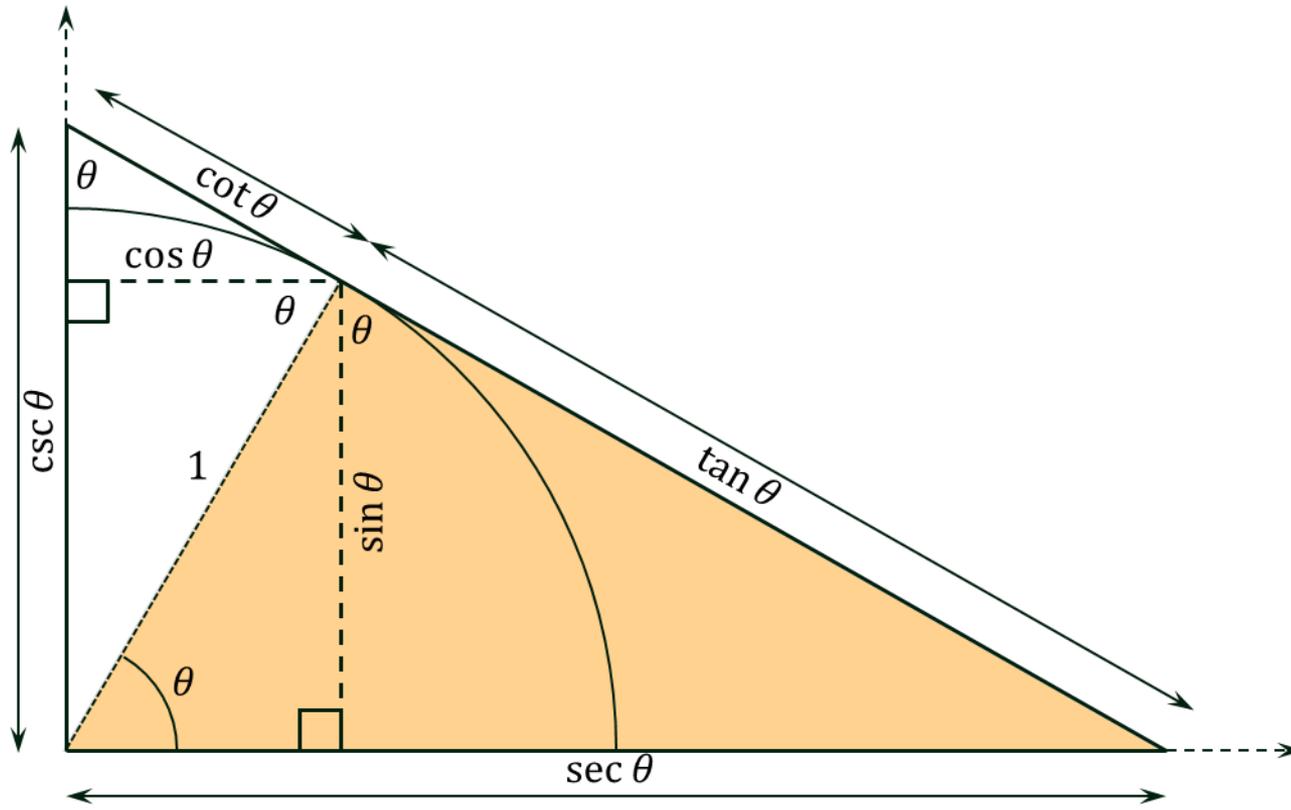
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



The SEC Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

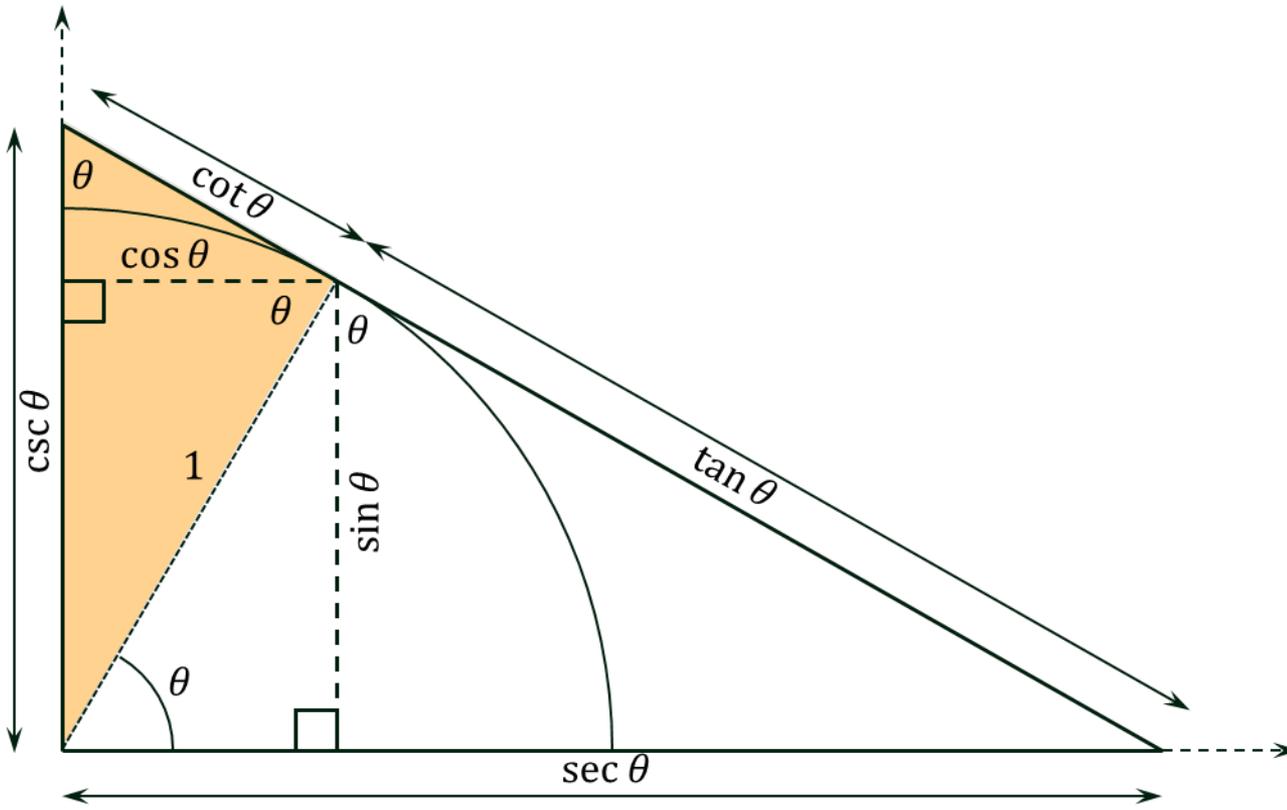
$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



The CSC Function



$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\csc(\theta) = \frac{H}{O}$$



The Reciprocal Identities

$$\sin(\theta) = \frac{O}{H}$$

$$\csc(\theta) = \frac{H}{O}$$

$$\csc(\theta) = \frac{1}{\sin(\theta)}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\sec(\theta) = \frac{1}{\cos(\theta)}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\cot(\theta) = \frac{1}{\tan(\theta)}$$



The Ratio Identities

$$\sin(\theta) = \frac{O}{H}$$

$$\csc(\theta) = \frac{H}{O}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\sec(\theta) = \frac{H}{A}$$

$$\tan(\theta) = \frac{O}{A}$$

$$\cot(\theta) = \frac{A}{O}$$

$$\tan(\theta) = \frac{\sin(\theta)}{\cos(\theta)}$$

$$\cot(\theta) = \frac{\cos(\theta)}{\sin(\theta)}$$

$$\tan(\theta) = \frac{\sec(\theta)}{\csc(\theta)}$$

$$\cot(\theta) = \frac{\csc(\theta)}{\sec(\theta)}$$



The Pythagorean Identities

$$\sin(\theta) = \frac{O}{H}$$

$$\cos(\theta) = \frac{A}{H}$$

$$\sin^2(\theta) + \cos^2(\theta) = 1$$

$$\frac{\sin^2(\theta)}{\cos^2(\theta)} + \frac{\cos^2(\theta)}{\cos^2(\theta)} = \frac{1}{\cos^2(\theta)}$$

$$\tan^2(\theta) + 1 = \sec^2(\theta)$$

$$\frac{\sin^2(\theta)}{\sin^2(\theta)} + \frac{\cos^2(\theta)}{\sin^2(\theta)} = \frac{1}{\sin^2(\theta)}$$

$$1 + \cot^2(\theta) = \csc^2(\theta)$$



The Symmetry Identities

The graph of $\sin(\theta)$ is symmetric about the origin,
therefore...

$$\sin(-\theta) = -\sin(\theta)$$

The graph of $\cos(\theta)$ is symmetric about the y-axis,
therefore...

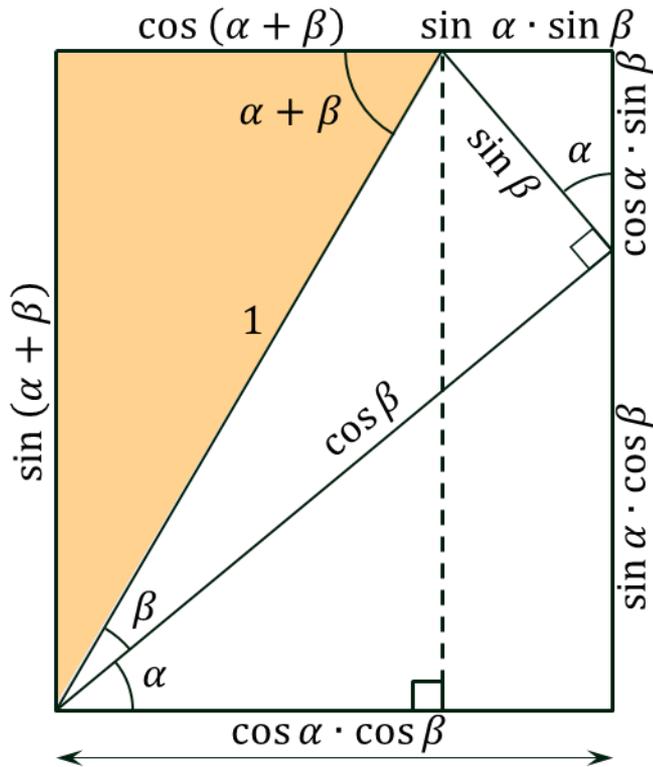
$$\cos(-\theta) = \cos(\theta)$$

The graph of $\tan(\theta)$ is symmetric about the origin,
therefore...

$$\tan(-\theta) = -\tan(\theta)$$



SIN of Sums (and Differences)



$$\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$\sin(-\beta) = -\sin \beta$$

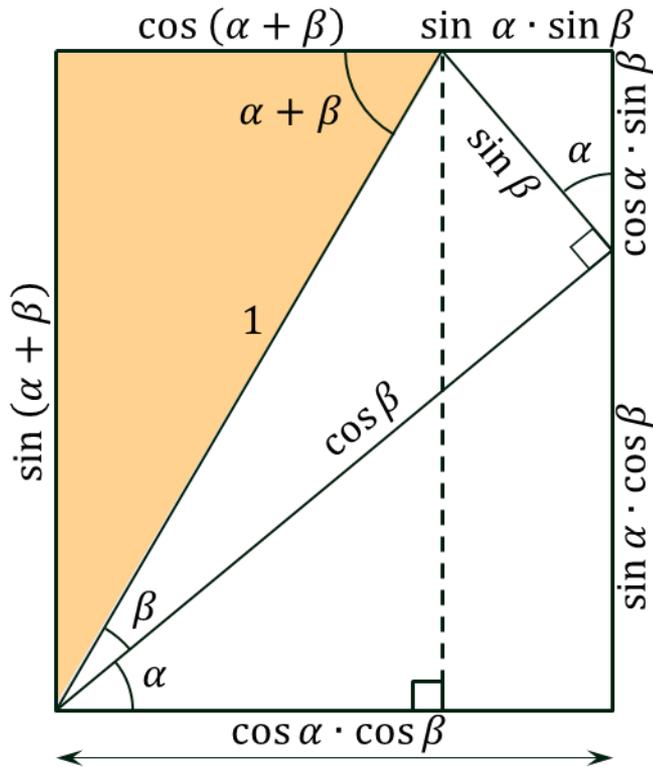
$$\cos(-\beta) = \cos \beta$$

$$\begin{aligned} \sin(\alpha - \beta) &= \sin(\alpha + (-\beta)) \\ &= \sin \alpha \cdot \cos(-\beta) + \cos \alpha \cdot \sin(-\beta) \end{aligned}$$

$$\sin(\alpha - \beta) = \sin \alpha \cdot \cos \beta - \cos \alpha \cdot \sin \beta$$



COS of Sums (and Differences)



$$\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$\sin(-\beta) = -\sin \beta$$

$$\cos(-\beta) = \cos \beta$$

$$\begin{aligned} \cos(\alpha - \beta) &= \cos(\alpha + (-\beta)) \\ &= \cos \alpha \cdot \cos(-\beta) - \sin \alpha \cdot \sin(-\beta) \end{aligned}$$

$$\cos(\alpha - \beta) = \cos \alpha \cdot \cos \beta + \sin \alpha \cdot \sin \beta$$



Double Angle Formulae

$$\sin(\alpha + \beta) = \sin \alpha \cdot \cos \beta + \cos \alpha \cdot \sin \beta$$

$$\sin(2\theta) = \sin(\theta + \theta) = \sin \theta \cdot \cos \theta + \cos \theta \cdot \sin \theta$$

$$\sin(2\theta) = 2 \cdot \sin \theta \cdot \cos \theta$$

$$\cos(\alpha + \beta) = \cos \alpha \cdot \cos \beta - \sin \alpha \cdot \sin \beta$$

$$\cos(2\theta) = \cos(\theta + \theta) = \cos \theta \cdot \cos \theta - \sin \theta \cdot \sin \theta$$

$$\cos(2\theta) = \cos^2 \theta - \sin^2 \theta$$

$$\cos(2\theta) = (1 - \sin^2 \theta) - \sin^2 \theta = 1 - 2 \cdot \sin^2 \theta$$

$$\cos(2\theta) = \cos^2 \theta - (1 - \cos^2 \theta) = 2 \cdot \cos^2 \theta - 1$$



Double Angle Formulae

$$\tan(\alpha + \beta) = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \cdot \tan \beta}$$

$$\tan(2\theta) = \tan(\theta + \theta) = \frac{\tan \theta + \tan \theta}{1 - \tan \theta \cdot \tan \theta}$$

$$\tan(2\theta) = \frac{2 \cdot \tan \theta}{1 - \tan^2 \theta}$$

$$\frac{2 \cdot \tan \theta}{1 - \tan^2 \theta} \cdot \frac{\cos^2(\theta)}{\cos^2(\theta)} = \tan(2\theta) = \frac{\sin(2\theta)}{\cos(2\theta)} = \frac{2 \cdot \sin \theta \cdot \cos \theta}{\cos^2 \theta - \sin^2 \theta}$$



Half Angle Formulae

$$\cos(2\theta) = 1 - 2 \cdot \sin^2\theta$$

$$2 \cdot \sin^2\theta = 1 - \cos(2\theta)$$

$$\sin^2\theta = \frac{1 - \cos(2\theta)}{2}$$

$$\sin \theta = \pm \sqrt{\frac{1 - \cos(2\theta)}{2}}$$

$$\sin \frac{\phi}{2} = \pm \sqrt{\frac{1 - \cos(\phi)}{2}}$$

$$\cos(2\theta) = 2 \cdot \cos^2\theta - 1$$

$$2 \cdot \cos^2\theta = 1 + \cos(2\theta)$$

$$\cos^2\theta = \frac{1 + \cos(2\theta)}{2}$$

$$\cos \theta = \pm \sqrt{\frac{1 + \cos(2\theta)}{2}}$$

$$\cos \frac{\phi}{2} = \pm \sqrt{\frac{1 + \cos(\phi)}{2}}$$

$$\frac{\phi}{2} = \theta$$



Half Angle Formulae

$$\tan \frac{\phi}{2} = \frac{\sin \frac{\phi}{2}}{\cos \frac{\phi}{2}} = \frac{\pm \sqrt{\frac{1 - \cos(\phi)}{2}}}{\pm \sqrt{\frac{1 + \cos(\phi)}{2}}}$$

$$\tan \frac{\phi}{2} = \pm \sqrt{\frac{1 - \cos(\phi)}{1 + \cos(\phi)}}$$

$$\tan \frac{\phi}{2} = \frac{\sqrt{1 - \cos(\phi)}}{\sqrt{1 + \cos(\phi)}} \cdot \frac{\sqrt{1 + \cos(\phi)}}{\sqrt{1 + \cos(\phi)}} = \frac{\sqrt{1 - \cos^2(\phi)}}{1 + \cos(\phi)}$$

$$= \frac{\sin(\phi)}{1 + \cos(\phi)}$$

$$\tan \frac{\phi}{2} = \frac{\sqrt{1 - \cos(\phi)}}{\sqrt{1 + \cos(\phi)}} \cdot \frac{\sqrt{1 - \cos(\phi)}}{\sqrt{1 - \cos(\phi)}} = \frac{1 - \cos(\phi)}{\sqrt{1 - \cos^2(\phi)}}$$

$$= \frac{1 - \cos(\phi)}{\sin(\phi)}$$



Plane Trigonometry

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“LAW of SINES”

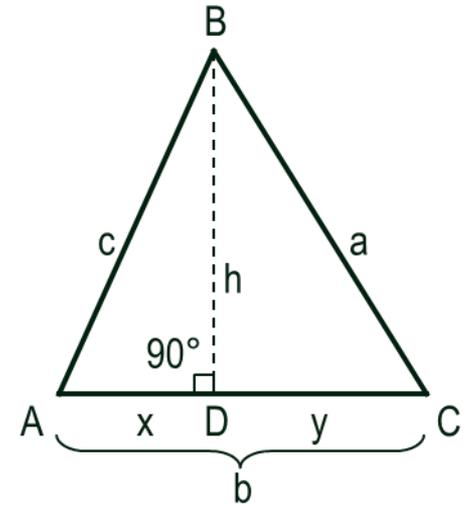
✔ **Equation 1** $h = c \cdot \sin A$

✔ **Equation 2** $h = a \cdot \sin C$

Equate the right sides of equations 1 and 2 and rearrange ...

✔ **Equation 3** $\frac{a}{\sin A} = \frac{c}{\sin C}$

Similarly: $\frac{a}{\sin A} = \frac{b}{\sin B}$ and $\frac{b}{\sin B} = \frac{c}{\sin C}$



“LAW of SINES”

$$\boxed{\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}} \quad \dots \text{ or } \dots \quad \boxed{\frac{\sin A}{a} = \frac{\sin B}{b} = \frac{\sin C}{c}}$$



“LAW of COSINES”

✔ **Equation 1** $h^2 = c^2 - x^2$

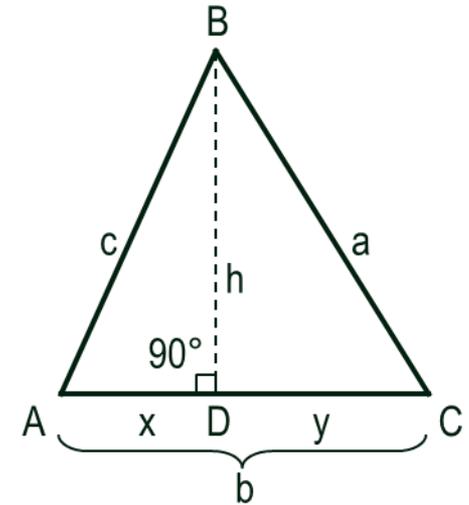
✔ **Equation 2** $h^2 = a^2 - y^2 = a^2 - (b - x)^2$

Equate the right sides of equations 1 and 2 and rearrange ...

$$a^2 - (b - x)^2 = c^2 - x^2$$

✔ **Equation 3** $a^2 = b^2 + c^2 - 2bx$

✔ **Equation 4** $x = c \cos A$



Substitute for x from equation 4 into equation 3...

“LAW of COSINES” $a^2 = b^2 + c^2 - 2bc \cos A$

Similarly: {

$$b^2 = a^2 + c^2 - 2ac \cos B$$

$$c^2 = a^2 + b^2 - 2ab \cos C$$

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}$$



Plane Triangle Areas

$$\text{Area } ABD = \text{Area } ABE = \frac{x \cdot h}{2}$$

$$\text{Area } CBD = \text{Area } CBF = \frac{y \cdot h}{2}$$

✓ Area of Triangle ABC...

$$\text{Area} = \frac{b \cdot h}{2}$$

$$h = a \sin C$$

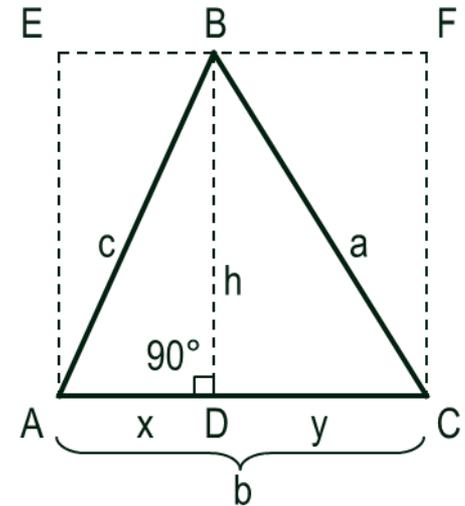
✓ Area of Triangle ABC...

$$\text{Area} = \frac{a \cdot b \cdot \sin C}{2}$$

$$b = \frac{a \cdot \sin B}{\sin A}$$

✓ Area of Triangle ABC...

$$\text{Area} = \frac{a^2 \cdot \sin B \cdot \sin C}{2 \cdot \sin A}$$





Spherical Trigonometry

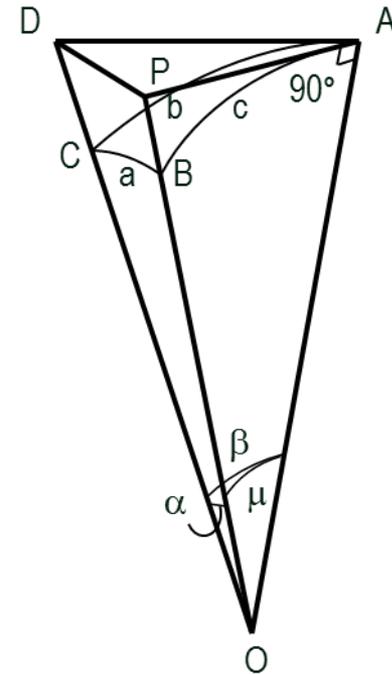
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The Spherical Trig “*LAW*S”

- ▼ *Law of Cosines*
- ▼ *Law of Sines*
- ▼ *Law of Cotangents*
- ▼ *Spherical Excess*

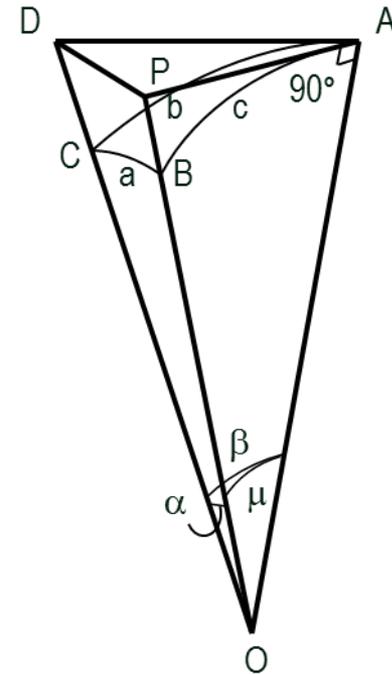




Basic Assumptions

Sides AD and AP are tangent at A
such that ...

- $DAP = \text{spherical angle } A$
- $DAO = 90^\circ$
- $PAO = 90^\circ$





Plane Trig Formulae

In plane triangle DPO ...

✔ **Equation 1**

$$DP^2 = DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos \alpha$$

In plane triangle DPA ...

✔ **Equation 2**

$$DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

In plane triangle ADO ...

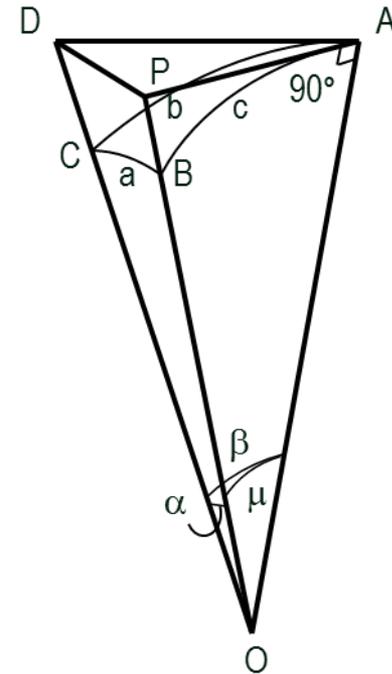
✔ **Equation 3**

$$DO^2 = AD^2 + AO^2$$

In plane triangle APO ...

✔ **Equation 4**

$$PO^2 = AP^2 + AO^2$$





✔ **Equation 1**

$$DP^2 = DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos \alpha$$

✔ **Equation 2**

$$DP^2 = AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

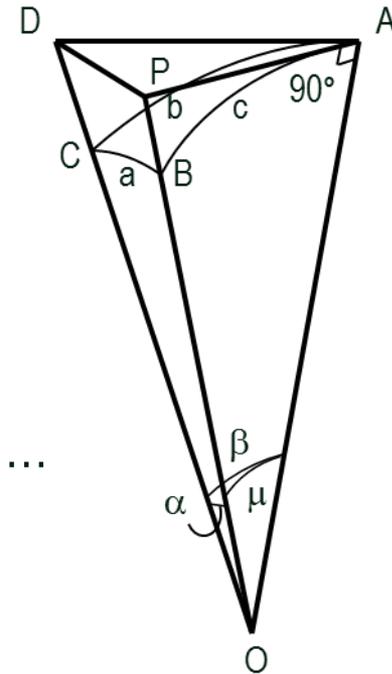
Equate the right sides of equations 1 and 2 and rearrange ...

$$DO^2 + PO^2 - 2 \cdot DO \cdot PO \cdot \cos \alpha =$$

$$AD^2 + AP^2 - 2 \cdot AD \cdot AP \cdot \cos A$$

✔ **Equation 5**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot DO \cdot PO \cdot \cos \alpha - 2 \cdot AD \cdot AP \cdot \cos A$$





✔ **Equation 3**

$$DO^2 = AD^2 + AO^2$$

✔ **Equation 4**

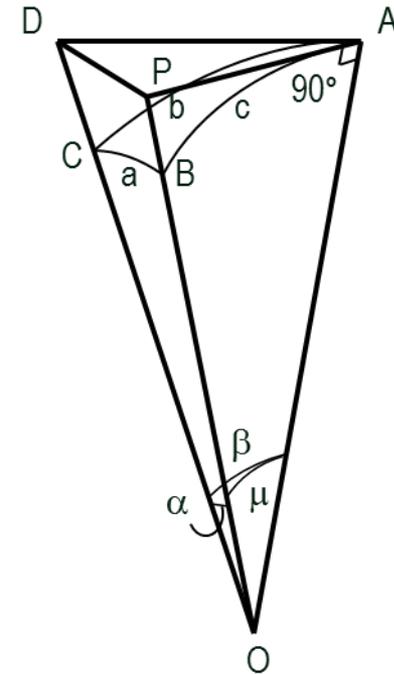
$$PO^2 = AP^2 + AO^2$$

Add equations 3 and 4 and rearrange ...

$$DO^2 + PO^2 = AD^2 + AP^2 + AO^2 + AO^2$$

✔ **Equation 6**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot AO^2$$





✔ **Equation 5**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot DO \cdot PO \cdot \cos \alpha - 2 \cdot AD \cdot AP \cdot \cos A$$

✔ **Equation 6**

$$DO^2 + PO^2 - AD^2 - AP^2 = 2 \cdot AO^2$$

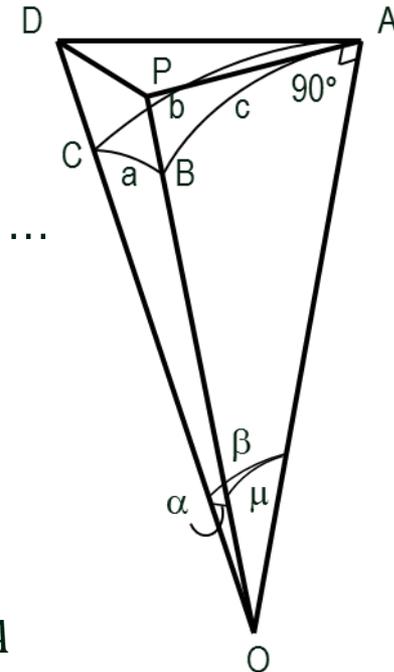
Equate the right sides of equations 5 and 6 and rearrange ...

$$2 \cdot DO \cdot PO \cdot \cos \alpha - 2 \cdot AD \cdot AP \cdot \cos A = 2 \cdot AO^2$$

$$DO \cdot PO \cdot \cos \alpha = AO \cdot AO + AD \cdot AP \cdot \cos A$$

$$\cos \alpha = \frac{AO \cdot AO}{DO \cdot PO} + \frac{AD \cdot AP}{DO \cdot PO} \cos A$$

$$\cos \alpha = \cos \beta \cdot \cos \mu + \sin \beta \cdot \sin \mu \cdot \cos A$$





“LAW of COSINES”

$$\cos \alpha = \cos \beta \cdot \cos \mu + \sin \beta \cdot \sin \mu \cdot \cos A$$

Note ... $\alpha = a$ $\beta = b$ $\mu = c$

“LAW of COSINES”

✔ Equation 7

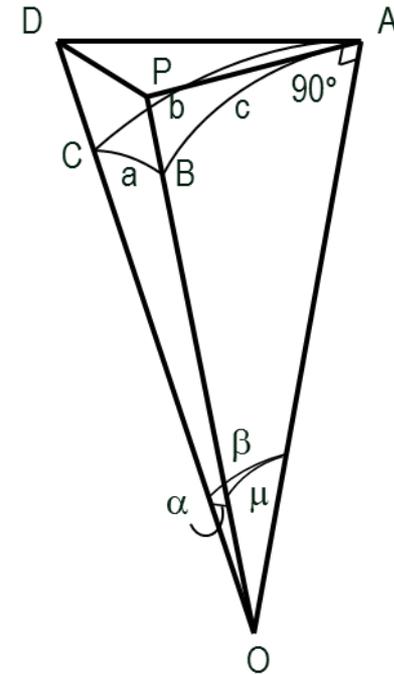
$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

✔ Equation 8

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$$

✔ Equation 9

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$





Rearrange equations 7, 8 and 9 ...

✔ **Equation 10** (from equation 7)

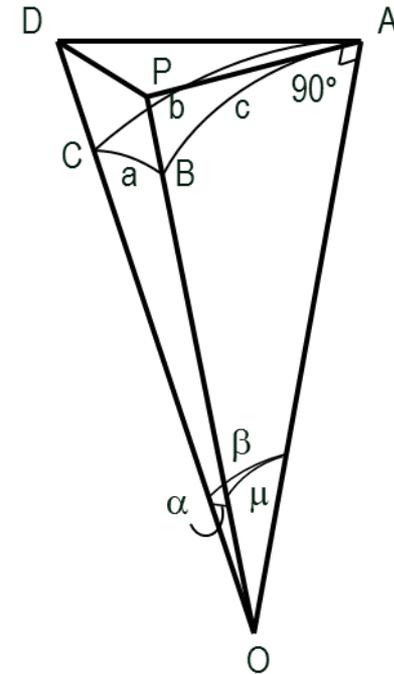
$$\cos a - \cos b \cdot \cos c = \sin b \cdot \sin c \cdot \cos A$$

✔ **Equation 11** (from equation 8)

$$\cos b - \cos a \cdot \cos c = \sin a \cdot \sin c \cdot \cos B$$

✔ **Equation 12** (from equation 9)

$$\cos c - \cos a \cdot \cos b = \sin a \cdot \sin b \cdot \cos C$$





✓ **Equation 10** (from equation 7)

$$\cos a - \cos b \cdot \cos c = \sin b \cdot \sin c \cdot \cos A$$

Square both sides of equation 10

$$\cos^2 a - 2 \cdot \cos a \cdot \cos b \cdot \cos c + \cos^2 b \cdot \cos^2 c = \sin^2 b \cdot \sin^2 c \cdot \cos^2 A$$

Apply the trig identity ... $\cos^2 \theta = 1 - \sin^2 \theta$

$$(1 - \sin^2 a) - 2 \cos a \cos b \cos c + (1 - \sin^2 b)(1 - \sin^2 c) = \sin^2 b \sin^2 c (1 - \sin^2 A)$$

$$1 - \sin^2 a - 2 \cos a \cos b \cos c + 1 - \sin^2 b - \sin^2 c + \sin^2 b \sin^2 c =$$

$$\sin^2 b \sin^2 c - \sin^2 b \sin^2 c \sin^2 A$$

✓ **Equation 13** (from equation 10)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 b \cdot \sin^2 c \cdot \sin^2 A$$



“LAW of SINES”

✓ **Equation 13** (from equation 10)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 b \cdot \sin^2 c \cdot \sin^2 A$$

✓ **Equation 14** (from equation 11)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 a \cdot \sin^2 c \cdot \sin^2 B$$

✓ **Equation 15** (from equation 12)

$$2 - \sin^2 a - \sin^2 b - \sin^2 c - 2 \cdot \cos a \cdot \cos b \cdot \cos c = -\sin^2 a \cdot \sin^2 b \cdot \sin^2 C$$

Equate the right sides of equations 13, 14 and 15 and rearrange ...

$$-\sin^2 b \cdot \sin^2 c \cdot \sin^2 A = -\sin^2 a \cdot \sin^2 c \cdot \sin^2 B = -\sin^2 a \cdot \sin^2 b \cdot \sin^2 C$$

$$\sin b \cdot \sin c \cdot \sin A = \sin a \cdot \sin c \cdot \sin B = \sin a \cdot \sin b \cdot \sin C$$

“LAW of SINES”

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}$$



✔ **Equation 16** (Law of Sines)

$$\sin b \cdot \sin A = \sin a \cdot \sin B$$

✔ **Equation 17** (Law of Cosines)

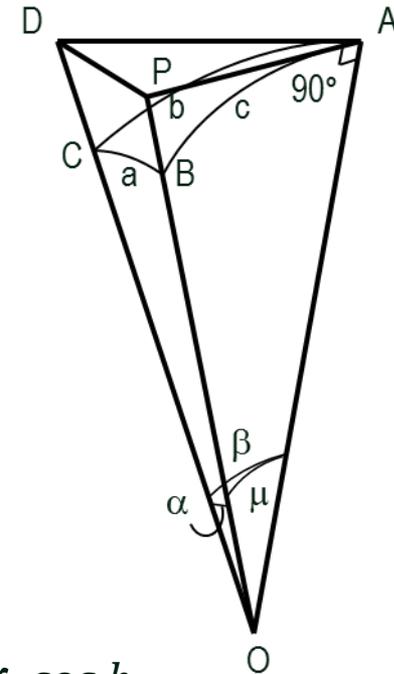
$$\cos a = \cos b \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

✔ **Equation 18** (Law of Cosines)

$$\cos b = \cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B$$

✔ **Equation 19** (Law of Cosines)

$$\cos c = \cos a \cdot \cos b + \sin a \cdot \sin b \cdot \cos C$$



Substitute the right side of equation 18 into equation 17 for $\cos b$

$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$



$$\cos a = (\cos a \cdot \cos c + \sin a \cdot \sin c \cdot \cos B) \cdot \cos c + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a = \cos a \cdot \cos^2 c + \sin a \cdot \sin c \cdot \cos c \cdot \cos B + \sin b \cdot \sin c \cdot \cos A$$

$$\cos a - \cos a \cdot \cos^2 c = \sin c \cdot \sin a \cdot \cos c \cdot \cos B + \sin c \cdot \sin b \cdot \cos A$$

$$\cos a \cdot (1 - \cos^2 c) = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

Apply the trig identity ... $\sin^2 \theta = 1 - \cos^2 \theta$

$$\cos a \cdot \sin^2 c = \sin c \cdot (\sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A)$$

$$\cos a \cdot \sin c = \sin a \cdot \cos c \cdot \cos B + \sin b \cdot \cos A$$

✓ **Equation 20**

$$\sin b \cdot \cos A = \cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B$$



“LAW of COTANGENTS”

✔ Equation 20

$$\sin b \cdot \cos A = \cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B$$

✔ Equation 16

$$\sin b \cdot \sin A = \sin a \cdot \sin B$$

Divide equation 16 into equation 20 and rearrange ...

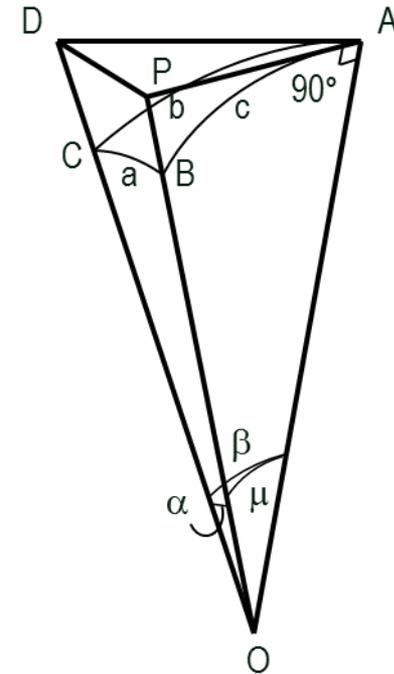
$$\frac{\sin b \cdot \cos A}{\sin b \cdot \sin A} = \frac{\cos a \cdot \sin c - \sin a \cdot \cos c \cdot \cos B}{\sin a \cdot \sin B}$$

“LAW of COTANGENTS”

$$\cot A = \frac{\cot a \cdot \sin c - \cos c \cdot \cos B}{\sin B}$$

... or ...

$$\tan A = \frac{\sin B}{\cot a \cdot \sin c - \cos c \cdot \cos B}$$





"SPHERICAL EXCESS"

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$$E = A + B + C - 180^\circ$$

$$\tan\left(\frac{E}{4}\right) = \sqrt{\tan\left(\frac{s}{2}\right) \cdot \tan\left(\frac{s-a}{2}\right) \cdot \tan\left(\frac{s-b}{2}\right) \cdot \tan\left(\frac{s-c}{2}\right)}$$

... where ...

$$s = \frac{a + b + c}{2}$$